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by

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The Generalized Cayley-Hamilton Theorem in n Dimensions

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In 1945, Reiner[(1)], by means of the Cayley-Hamilton theorem, obtained a canonical form for a polynomial relation between a stress matrix and a strain-velocity matrix; since that time an extensive theory has been developed for canonical forms of non-linear constitutive equations. More recently, for a polynomial relation between one tensor and a number of other tensors, the problem of finding the restrictions imposed by a symmetry group was reduced by Smith and Rivlin[(2)], and by Pipkin and Rivlin[(3)], to that of finding an integrity basis for a set of such tensors; and then, for the full (or proper) orthogonal group in Euclidean 2-space or 3-space, such a basis was determined by Rivlin[(4)], Spencer and Rivlin[(5)], and Spencer[(6)], and its irreducibility proven by Smith[(7)].

In this development, an important tool has been a generalization of the Cayley-Hamilton theorem, in 2-space or 3-space, from one to several matrix variables[(8)]. During this time, it has been clear that the corresponding identity in n-space, for any particular n, could be obtained in a finite but discouraging number of steps; however the form of this relation for an arbitrary n has not been given. We shall obtain this form, which is the intuitive generalization of the results in two and three

(1) See Reference 4.
(2) See Reference 7.
(3) See Reference 3.
(4) See Reference 5.
(5) See References 9,10,11.
(6) See Reference 8.
(7) See Reference 6.
(8) See Reference 5.

dimensions, and note that, properly expressed, it is a polynomial relation in the given matrices and their traces with all coefficients ± 1.

For an arbitrary real or complex n×n matrix A the Cayley-Hamilton theorem states that

$$1) \qquad \sum_{i=0}^{n} (-1)^i A^{n-i} s_i(A) = 0$$

where $A^0 = I$, and $s_i(A)$ is the i'th symmetric polynomial in the characteristic roots of A. If we let $t_j(A) = \operatorname{tr} A^j$ for $j = 1,2,\ldots$ then the well-known relations[9]

$$2) \qquad \begin{aligned} s_1 &= t_1 \\ 2s_2 &= s_1 t_1 - t_2 \\ 3s_3 &= s_2 t_1 - s_1 t_2 + t_3 \end{aligned}$$

and so forth can be solved recursively for each s_i in terms of $t_1, \ldots, t_i$ to yield

$$3) \qquad \begin{aligned} 1!s_1 &= t_1 \\ 2!s_2 &= t_1^2 - t_2 \\ 3!s_3 &= t_1^3 - 3t_1 t_2 + 2t_3 \end{aligned}$$

and so forth. Thus the Cayley-Hamilton theorem can be expressed as a relation in A and the $t_j(A)$.

In one dimension this process gives

$$4) \qquad A - I \operatorname{tr} A = 0$$

which is trivial; and in two dimensions it gives

$$5) \qquad A^2 - A \operatorname{tr} A + \tfrac{1}{2} I[(\operatorname{tr} A)^2 - \operatorname{tr} A^2] = 0.$$

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(9) See p. 9 of Reference 12.

If we apply to this equation the polarization operator d_{BA}, that is, if we replace A by A+xB, for a real variable x, and evaluate the derivative in x at the point x = 0, then we obtain

$$6)\quad AB + BA - A\,\mathrm{tr}\,B - B\,\mathrm{tr}\,A + I[\mathrm{tr}\,A\,\mathrm{tr}\,B - \mathrm{tr}\,AB] = 0$$

which is the generalized identity in two dimensions[10]. Note in this result that all permutations of A and B appear, since A and B need not commute, but that the fraction $\frac{1}{2}$ disappears, since scalars commute and traces have cyclical symmetry.

In three dimensions equations (1) and (3) give

$$7)\quad A^3 - A^2\mathrm{tr}\,A + \tfrac{1}{2}A[(\mathrm{tr}\,A)^2 - \mathrm{tr}\,A^2]$$
$$- \tfrac{1}{6}I[(\mathrm{tr}\,A)^3 - 3\,\mathrm{tr}\,A\,\mathrm{tr}\,A^2 + 2\,\mathrm{tr}\,A^3] = 0$$

applied to which the polarization operator d_{BA} again yields a relation in two variables. However, we desire a completely polarized relation, in which no matrix variable has degree more than unity. Thus if we also apply d_{CA} for another 3x3 matrix C, and let Σ denote the sum over all permutations of (A,B,C), then we obtain

$$8)\quad 0 = \Sigma ABC - \Sigma AB\,\mathrm{tr}\,C + \Sigma A[\mathrm{tr}\,B\,\mathrm{tr}\,C - \mathrm{tr}\,BC]$$
$$- I[\mathrm{tr}\,A\,\mathrm{tr}\,B\,\mathrm{tr}\,C - \mathrm{tr}\,A\,\mathrm{tr}\,BC - \mathrm{tr}\,B\,\mathrm{tr}\,CA - \mathrm{tr}\,C\,\mathrm{tr}\,AB$$
$$+ \mathrm{tr}\,ABC + \mathrm{tr}\,CBA]$$

which is the generalized identity in three dimensions[11]. Note here again that all fractions disappear by the properties of scalars and traces.

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(10) See Reference 5.
(11) See Reference 5.

Now in the derived expression for each s_i, the terms correspond to partitions of i, that is, to sequences $\mu = (m_1, m_2, ..)$ of non-negative integers with $q(\mu) = i$, where

9) $$p(\mu) = \sum_{j=1}^{\infty} (j-1)m_j \quad , \qquad q(\mu) = \sum_{j=1}^{\infty} jm_j \ .$$

Each m_j is interpreted as the number of subsets containing precisely j elements in the corresponding subdivision of a set containing precisely i elements, so that clearly $m_j = 0$ for $j > i$ and thus such sequences have all entries but a finite number equal to zero. If we let τ denote the sequence $(t_1, t_2, ...)$ with $t_j = \operatorname{tr} A^j$ as before, then we may let

10) $$\tau^\mu = t_1^{m_1} t_2^{m_2} ... \quad ,$$

a product which thus has all factors but a finite number equal to unity.

Each partition μ with $q(\mu) = q$ labels a conjugate class C in the group S_q containing all permutations of q elements, namely that class in which all permutations may be factored into disjoint cycles of which m_1 have length 1, m_2 have length 2, and so forth. The parity of all elements in C_μ is easily shown to be $\operatorname{sgn}(\mu) = (-1)^{p(\mu)}$, and the number of elements in C_μ is well-known to be[12]

11) $$c(\mu) = q!/m_1!m_2!...1^{m_1}2^{m_2}...$$

a quotient whose denominator has all factors but a finite number equal to unity. However, the general expression of the set (3) can then be written[13]

12) $$i!s_i = \sum_{q(\mu)=i} \operatorname{sgn}(\mu) c(\mu) \tau^\mu$$

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(12) See 3.6 of Reference 1 or IV.4 of Reference 2.
(13) See 6.2 of Reference 1 or (4.24) of Reference 2.

and the form (1) of the Cayley-Hamilton theorem can be rewritten

13) $$\sum_{i=0}^{n} (-1)^i A^{n-i} \sum_{q(\mu)=i} \operatorname{sgn}(\mu) c(\mu) \tau^{\mu}(A)/i! = 0.$$

Now we need only completely polarize this equation, noting that it is homogeneous of degree n in A; that is, we need only replace the n equal variables A by all permutations of n distinct variables $A_1,\ldots,A_n$, and put the sum of all such expressions equal to zero. For each i the corresponding term in (13) then yields n! terms, of which we may collect all those terms such that $A_{\pi(1)},\ldots,A_{\pi(i)}$, in any order, appear in the inner sum, and $A_{\pi(i+1)},\ldots,A_{\pi(n)}$, in any order, appear in the outer sum. Thus for each i we obtain a sum over the $\binom{n}{i}$ ways to select a subset of i elements from $\{A_1,\ldots,A_n\}$, with each summand of the form

14)
$$(-1)^i[\Sigma \text{ all permutations of } A_{\pi(i+1)}\cdots A_{\pi(u)}]\operatorname{coeff.}(A_{\pi(1)},\ldots,A_{\pi(i)})$$

and to find the coefficient for each selection we need only replace the i equal variables A by all permutations of i distinct variables $A_{\pi(1)},\ldots,A_{\pi(i)}$ in

15) $$\sum_{q(\mu)=i} \operatorname{sgn}(\mu)(\operatorname{tr} A)^{m_1}\cdots(\operatorname{tr} A^i)^{m_i}/m_1!\cdots m_i!1^{m_1}\cdots i^{m_i}$$

and take the sum of all such expressions.

But for each μ in the sum (15) many of the resulting i! terms are equal; in particular, for each j the m_j traces of products containing j factors may be permuted in all $m_j!$ ways without changing the result, and in each of these m_j traces the j factors may be cycled in j ways without changing the result. Since the various terms are otherwise distinct, the order of their

degeneracy is precisely the denominator associated with the given μ in the sum (15), and thus the coefficient is precisely the sum, over all partitions μ of i and all essentially distinct permutations of the $A_{\pi(j)}$, of terms

$$\text{16)}\quad \operatorname{sgn}(\mu) \prod_{j=1}^{m_1} \operatorname{tr}(A_{\pi(j)}) \prod_{j=1}^{m_2} \operatorname{tr}(A_{\pi(m_1+2j-1)}, A_{\pi(m_1+2j)}) \cdots .$$

In summary, the generalized Cayley-Hamilton theorem in n dimensions asserts the vanishing of the sum for $i = 0,\ldots,n$ of all essentially distinct terms of the form (14), in which the coefficient is the sum for $q(\mu) = i$ of all essentially distinct terms of the form (16). Furthermore, the coefficients are independent of n, and have the forms given in equation (8) for $i = 1,2,3$; finally, by the principle just stated, the coefficient for $i = 4$ has the form

17)

$$\begin{aligned}&\mathrm{trAtrBtrCtrD - trAtrBtrCD - trAtrCtrBD - trAtrDtrBC - trBtrCtrAD}\\ &\mathrm{- trBtrDtrAC - trCtrDtrAB + trA(trBCD + trDCB) + trB(trACD + trDCA)}\\ &\mathrm{+ trC(trABD + trDBA) + trD(trABC + trCBA) + trABtrCD + trACtrBD}\\ &\mathrm{+ trADtrBC - trABCD - trABDC - trACBD - trACDB - trADBC - trADCB}\end{aligned}$$

Since the polarization process d_{BA} can be defined over any field of characteristic zero[14], these results are all valid over any such field. Indeed since the coefficients are simpler in the completely polarized equations, these results may well be provable directly in n variables, rather than through (13). However, this discussion indicates the explicit form of the desired relation, and thus reduces the labor of deriving it to merely that of writing it down.

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(14) See p. 4 of Reference 12.

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For any positive integer n, this paper derives the explicit form of the identity in n matrices, each nxn, which is obtained by complete polarization of the usual Cayley/Hamiston theorem, and which is used repeatedly, for n = 2 or 3, in the determination of an integrity basis for symmetric tensors.

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